Session 5: Probability 2

Stats 60/Psych 10
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Summer 2020

News

- Probability Review - Tuesday 14th, 1:30PM PDT
- Problems already available on the course website
- Try to solve them before the review!

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- **Practice Problems are available** on the course website
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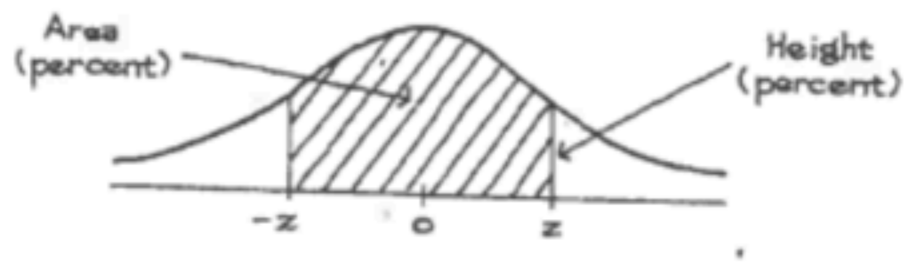
Last time

- What is a probability?
- Rules of probability
- Probability distributions

This time

- The normal probability distribution
- Conditional probability
- Bayes' rule

The normal distribution



A NORMAL TABLE

z	Height	Area	z	Height	Area	z	Height	Area
0.00	39.89	0	1.50	12.95	86.64	3.00	0.443	99.730
0.05	39.84	3.99	1.55	12.00	87.89	3.05	0.381	99.771
0.10	39.69	7.97	1.60	11.09	89.04	3.10	0.327	99.806
0.15	39.45	11.92	1.65	10.23	90.11	3.15	0.279	99.837
0.20	39.10	15.85	1.70	9.40	91.09	3.20	0.238	99.863
0.25	38.67	19.74	1.75	8.63	91.99	3.25	0.203	99.885
0.30	38.14	23.58	1.80	7.90	92.81	3.30	0.172	99.903
0.35	37.52	27.37	1.85	7.21	93.57	3.35	0.146	99.919
0.40	36.83	31.08	1.90	6.56	94.26	3.40	0.123	99.933
0.45	36.05	34.73	1.95	5.96	94.88	3.45	0.104	99.944
0.50	35.21	38.29	2.00	5.40	95.45	3.50	0.087	99.953
0.55	34.29	41.77	2.05	4.88	95.96	3.55	0.073	99.961
0.60	33.32	45.15	2.10	4.40	96.43	3.60	0.061	99.968
0.65	32.30	48.43	2.15	3.96	96.84	3.65	0.051	99.974
0.70	31.23	51.61	2.20	3.55	97.22	3.70	0.042	99.978

The normal distribution

- Normal table:
  - z-score
  - Height
  - Area

The normal distribution

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- **Learning Goals:**
  - derive percentiles from the table
  - understand why z-scores are useful

The normal distribution

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  - derive percentiles from the table
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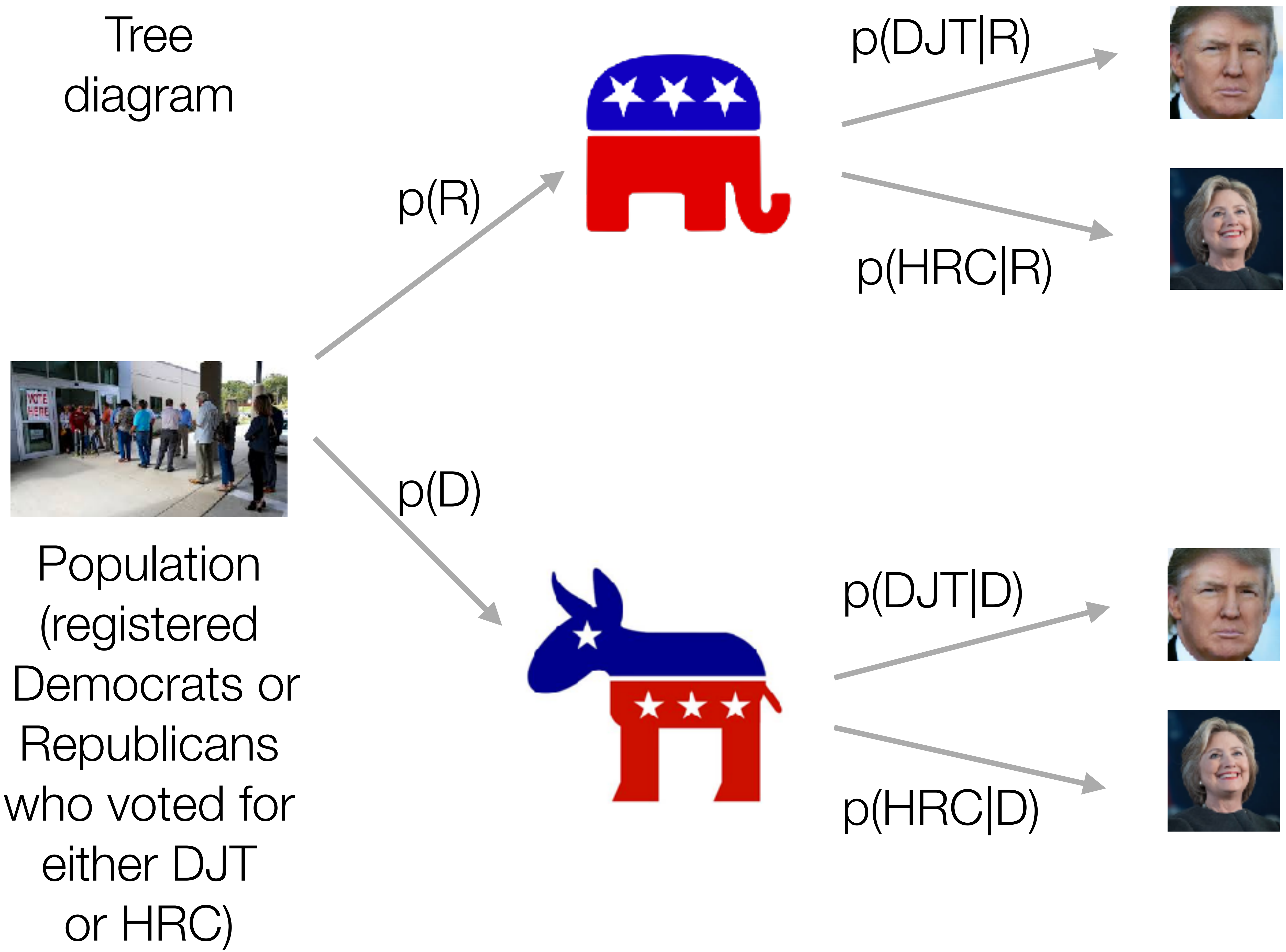
- https://shiny.rit.albany.edu/stat/stdnormal/
- More on this in Tuesday's review!

Conditional probability

- Simple probabilities:
  - What is the likelihood that a US voter was a Republican in 2016?
    - p(Republican) = 0.44
  - What is the likelihood that a US voter voted for Donald Trump in the 2016 Presidential Election?
    - P(TrumpVoter) = 0.46

Conditional probability

- Simple probabilities:
  - What is the likelihood that a US voter was a Republican in 2016?
    - p(Republican) = 0.44
  - What is the likelihood that a US voter voted for Donald Trump in the 2016 Presidential Election?
    - P(TrumpVoter) = 0.46
- Conditional probability: Probability of one event, given that some other has occurred
  - P(TrumpVoter|Republican) = ?


Tree diagram
p(R)
p(D)
p(DJT|R)
p(HRC|R)
p(DJT|D)
p(HRC|D)
Population (registered Democrats or Republicans who voted for either DJT or HRC)

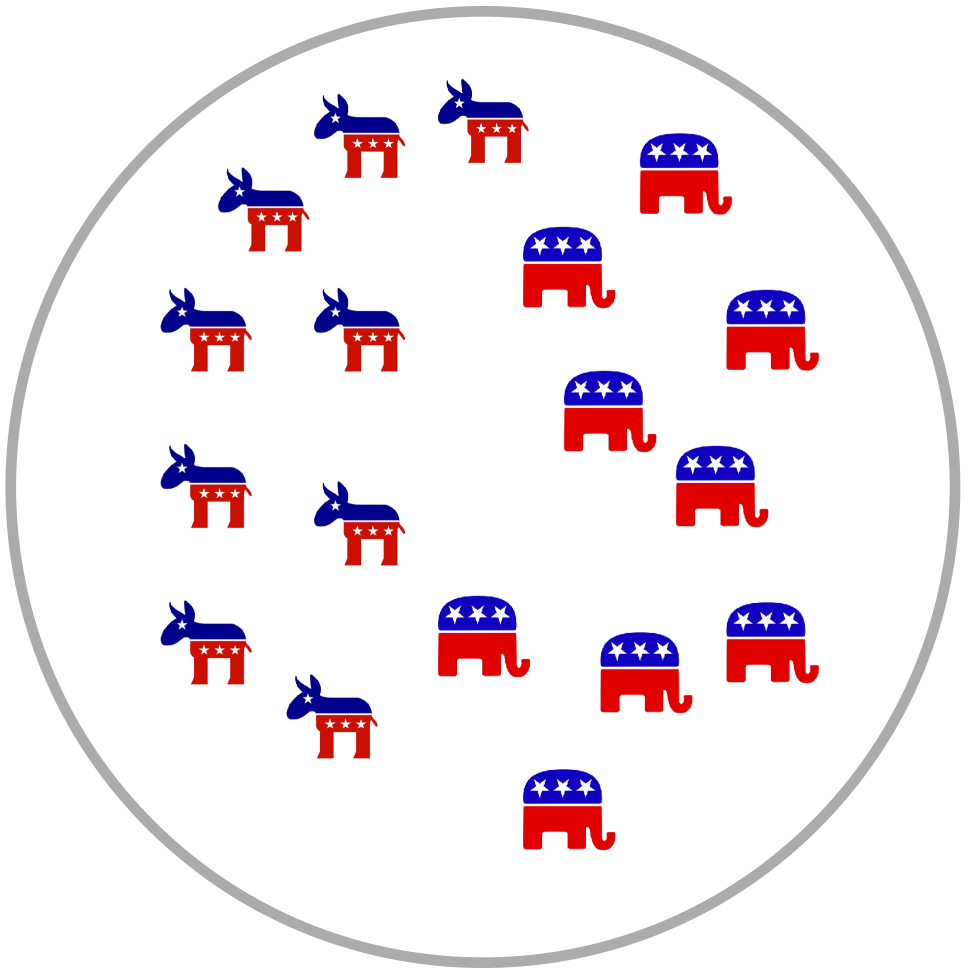
Computing conditional probability

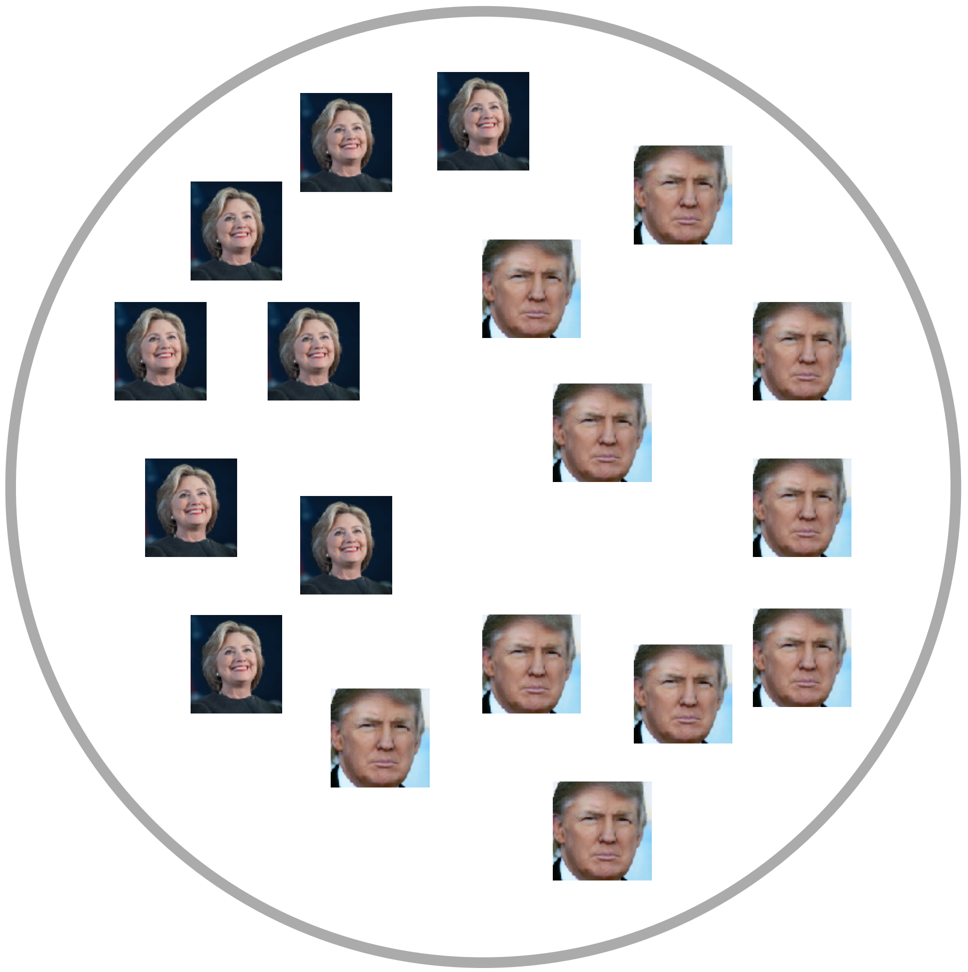
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(TrumpVoter|Republican) = \frac{P(TrumpVoter \cap Republican)}{P(Republican)}$$

Limits the calculation to the set of B events

Another view on conditional probability





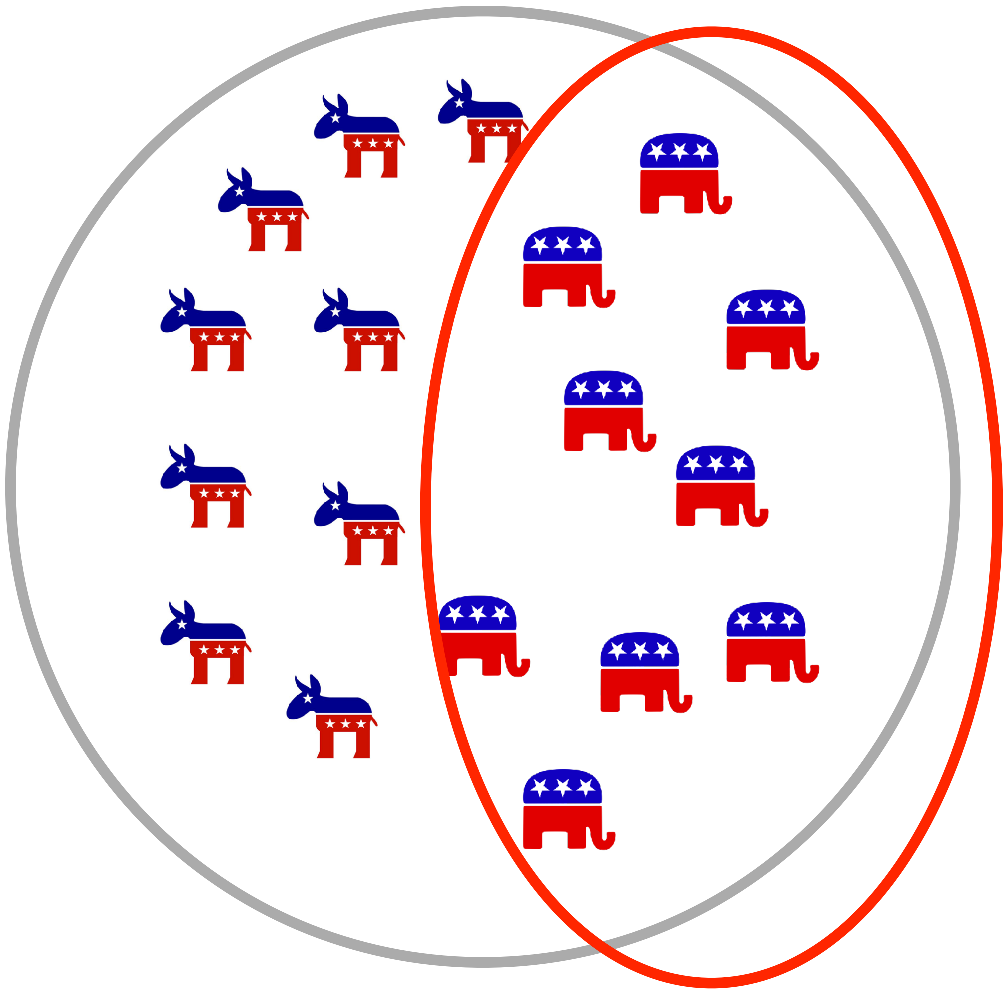
P(D)=9/18=0.5

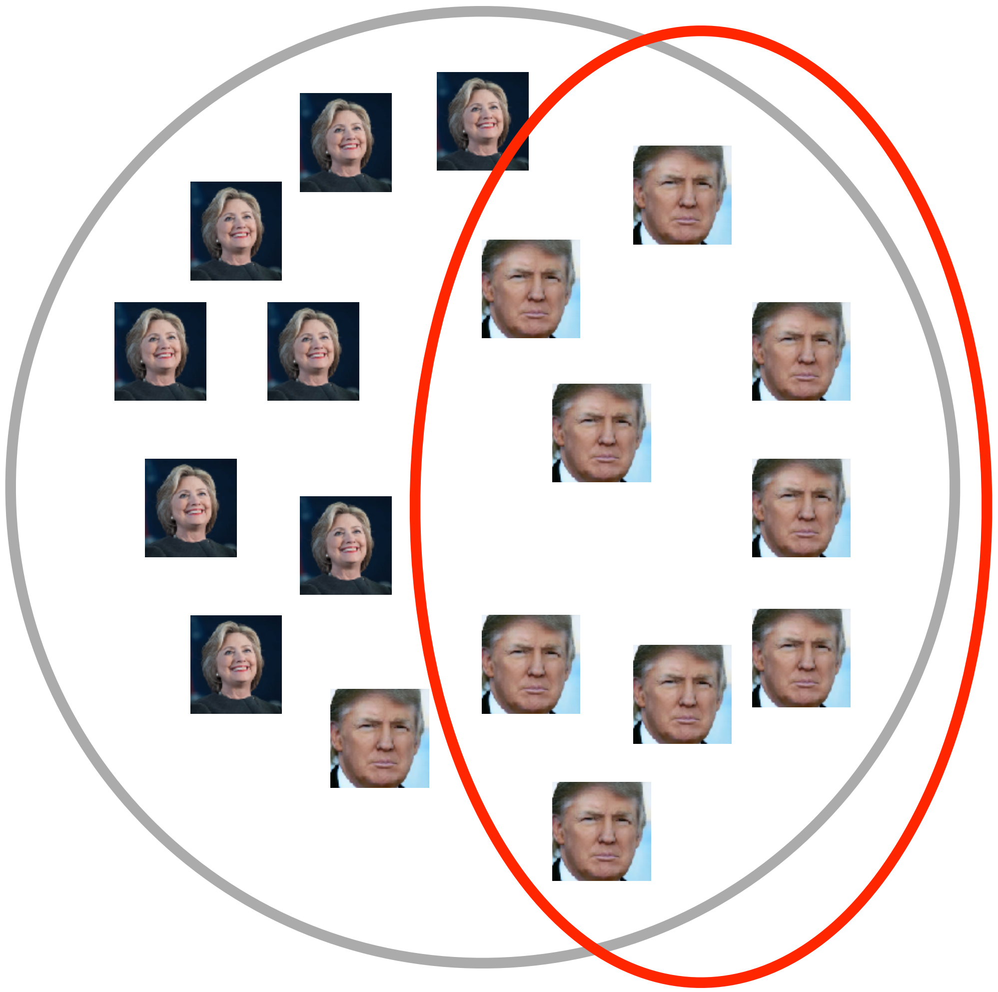
P(R) = 1 - P(D) = 0.5

P(DJT)=10/18=0.55

P(HRC) = 1 - P(DJT) = 0.45

Another view on conditional probability





$P(DJT)=10/18=0.55$

$P(DJT|R) = ?$

$P(DJT|R) = 9/9 = 1.0$

What does “independent” mean to you?

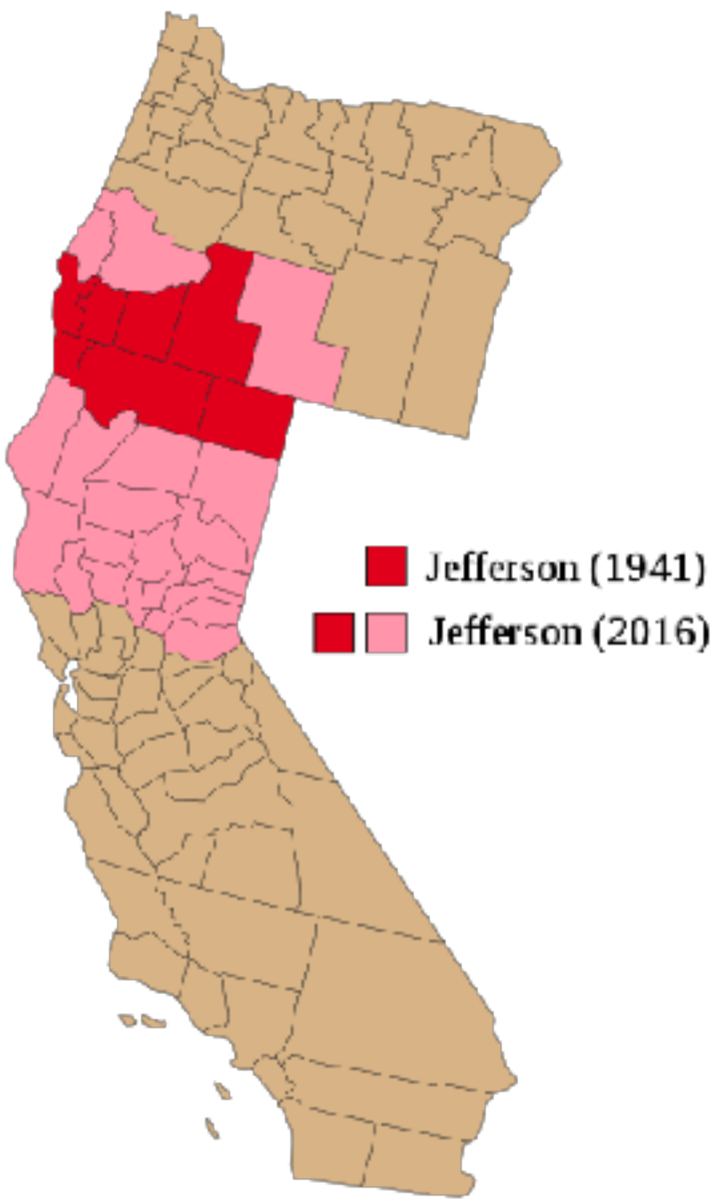
Statistical Independence

- Knowing about one thing does not tell us anything about the other

$$P(A|B) = P(A)$$

  - Knowing the value of B doesn't give us any additional information about the value of A
  - They are statistically unrelated
- This has a very different meaning from the common language meaning of "independence"

Example: The proposed “independent” state of Jefferson



Let’s suppose they succeeded
For a current resident of CA:

$P(CA)=0.986$

$P(JF)=0.014$

$P(CA|JF)=0$

political independence =
statistical dependence!

In general, mutually independent events will be statistically dependent (assuming $p>0$)

National Health and Nutrition Examination Survey

- NHANES is a program of studies by the CDC designed to assess the health and nutritional status of adults and children in the United States. The survey is unique in that it combines interviews and physical examinations.
- The survey examines a nationally representative sample of about 5,000 persons each year.
- The NHANES interview includes demographic, socioeconomic, dietary, and health-related questions. The examination component consists of medical, dental, and physiological measurements, as well as laboratory tests administered by highly trained medical personnel.
- Available in R:
  - `library(NHANES)`

An example: Are physical activity and mental health independent in NHANES?

PhysActive
Participant does moderate or vigorous-intensity sports, fitness or recreational activities (Yes or No).

DaysMentHlthBad
Self-reported number of days participant's mental health was not good out of the past 30 days.

```
NHANES_adult = NHANES_adult %>%
  mutate(badMentalHealth=DaysMentHlthBad>7)
```

Are two draws from a single deck of cards (without replacing the first card) independent?

yes **A**

no **B**

An example: Are physical activity and mental health independent in NHANES?

```
NHANES_adult %>%
 summarize(badMentalHealth=mean(badMentalHealth))
```

P(badMentalHealth)
0.164

```
NHANES_adult %>%
  group_by(PhysActive) %>%
  summarize(badMentalHealth=mean(badMentalHealth))
```

P(badMentalHealth|~Active)	0.200
P(badMentalHealth|Active)	0.132

Physical activity is good - let's do some!

Why independence matters

https://www.ted.com/talks/peter_donnelly_shows_how_stats_fool_juries

Reversing a conditional probability

- We known P(A|B)
- How do we find out what P(B|A) is?
  - Why would this ever be useful?

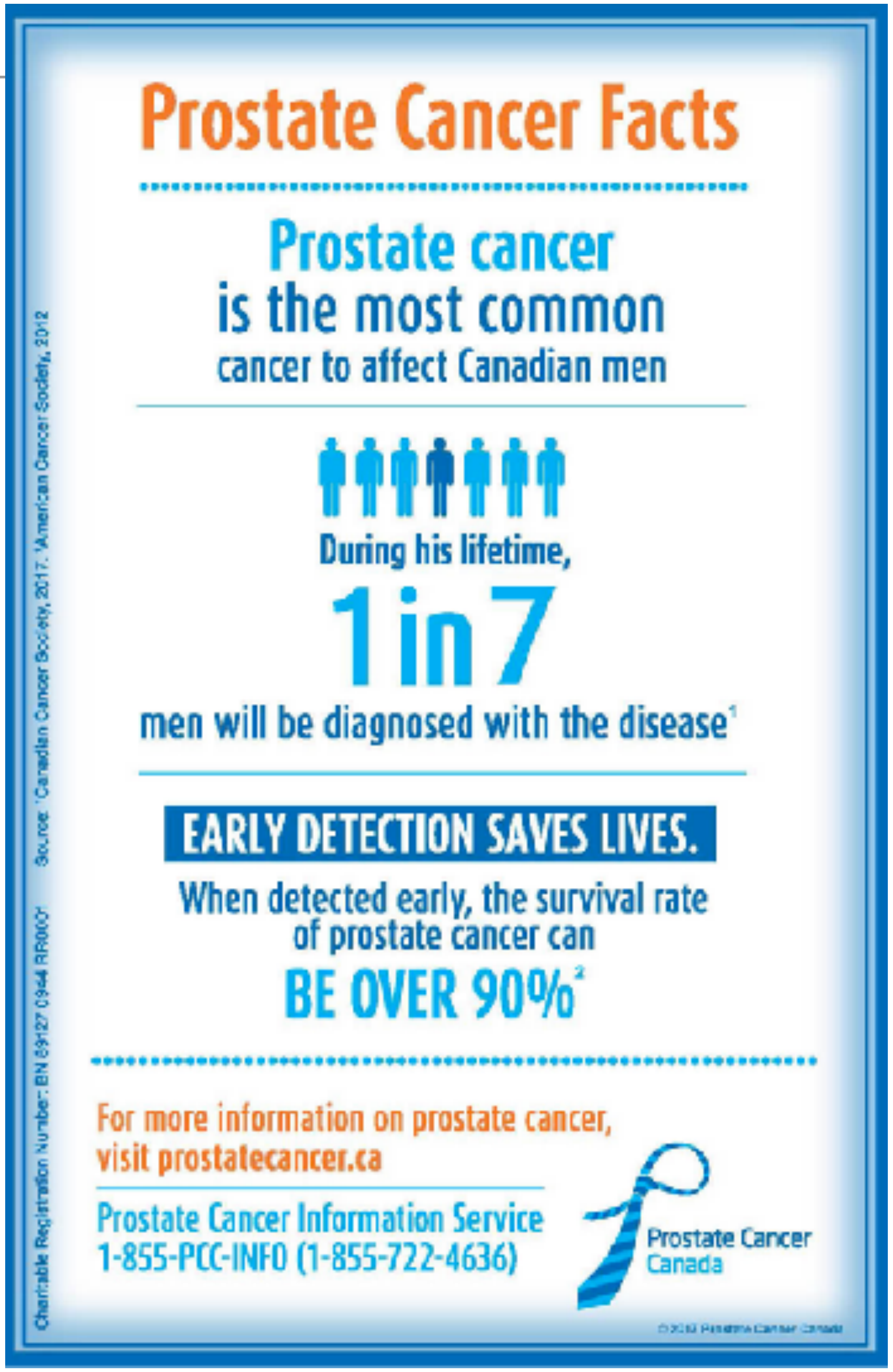
Airport screening

we know: P(positive test | explosives)

we want to know: P(explosives| positive test)

Medical testing

- Prostate specific antigen (PSA)
- Tests can be characterized by two factors:
    - Sensitivity:
        - P(positive test | disease)
        - ~80%
    - Specificity:
        - 1 - P(positive test| no disease)
        - ~70%



https://emedicine.medscape.com/article/457394-overview

Table of possible outcomes

	Has disease	Does not have disease
Positive test	"hit" $P(D \cap T)$	"false alarm" $P(\sim D \cap T)$
Negative test	"miss" $P(D \cap \sim T)$	"true negative" $P(\sim D \cap \sim T)$

Sensitivity: P(positive test | has disease)

How do we compute it?

Sensitivity = hits / (hits + misses)

Table of possible outcomes

	Has disease	Does not have disease
Positive test	"hit" $P(D \cap T)$	"false alarm" $P(\sim D \cap T)$
Negative test	"miss" $P(D \cap \sim T)$	"true negative" $P(\sim D \cap \sim T)$

Specificity: P(negative test | no disease)

How do we compute it?

Specificity = true negatives/(false alarms + true negatives)

A person selected at random receives a test for a disease and the result is positive. What do we need to know in order to determine the likelihood that the person actually has the disease? (select all that apply)

- The specificity of the test
- The sensitivity of the test
- The probability of getting the test
- The probability that the person has the disease

Interpreting test results

- A person receives a positive test result
- We know the likelihood of a positive test given the disease
  - Sensitivity of the test: P(positive test|disease)
- But what we really want to know is: is the likelihood that the person actually has the disease?
  - P(disease | positive test)

- How do we compute this “inverse probability”?

Bayes' rule

- A way to invert a conditional probability

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)}$$

- In the context of science:

$$P(hypothesis|data) = \frac{P(data|hypothesis)P(hypothesis)}{P(data)}$$

Deriving Bayes' rule

- Remember the definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- Rearrange to get the rule for computing joint probability of A and B:

$$P(A \cap B) = P(A|B)P(B)$$

- So if we want to compute P(B|A):

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$$

Bayes' rule

- For two outcomes, we can express it in a slightly clearer way using the sum rule for probabilities:

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)}$$

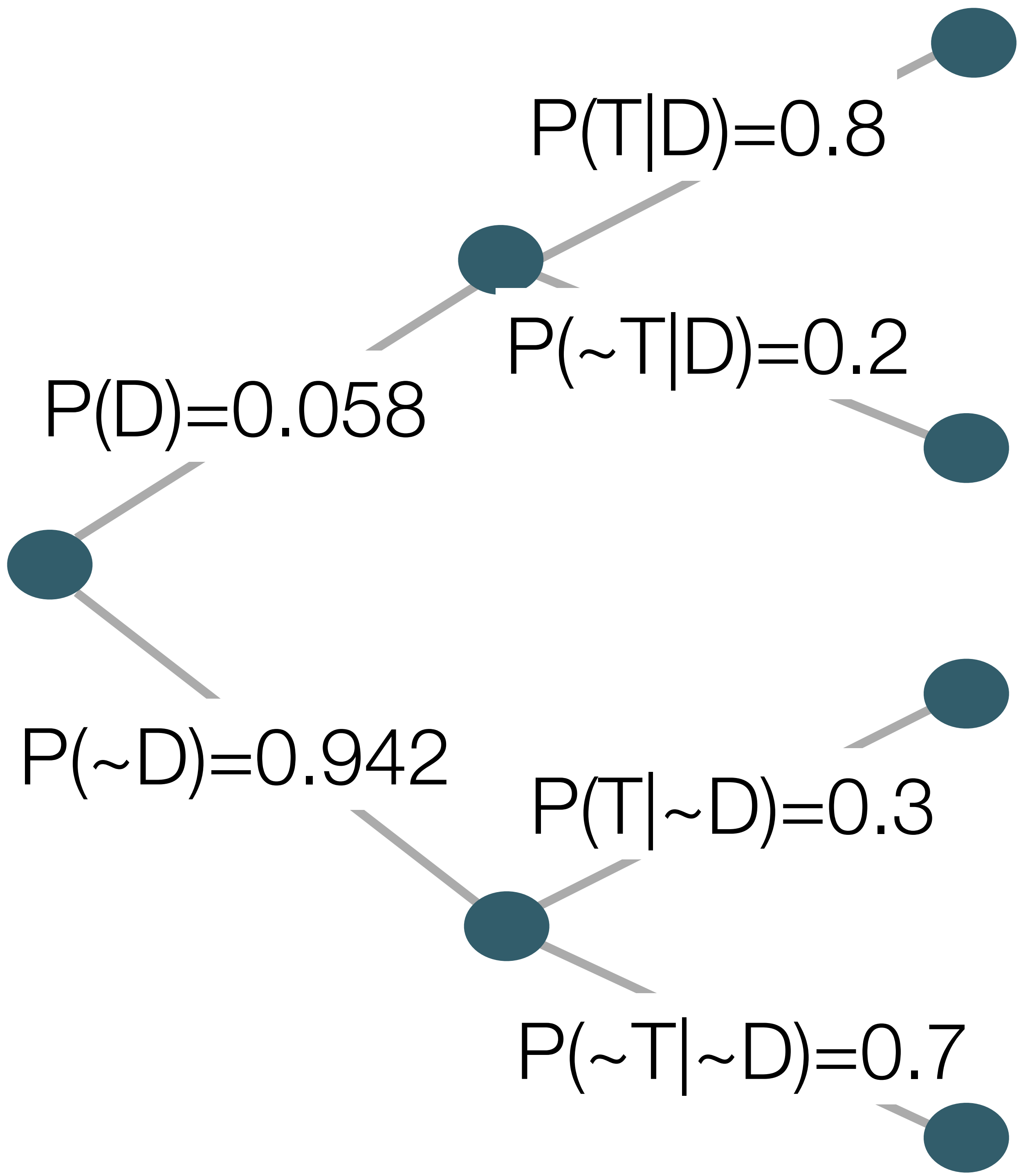
$$P(B) = P(B|A) * P(A) + P(B| \sim A) * P(\sim A)$$

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B|A) * P(A) + P(B| \sim A) * P(\sim A)}$$

60 year old male: P(disease in next 10 years)=0.058

Sensitivity: P(T|D)=0.8

Specificity: P(~T|~D)=0.7



$$P(D|T)=\frac{0.8*0.058}{0.8*0.058 + 0.3*0.942}$$

$$= 0.14$$

https://www.cdc.gov/cancer/prostate/statistics/age.htm

What do these probabilities mean?

- The person either has a disease or doesn't
- How should we interpret this probability?
- Objective probability
  - long-run relative frequency that the hypothesis is true
- Subjective probability
  - our degree of belief in the hypothesis
  - how plausible is the hypothesis?

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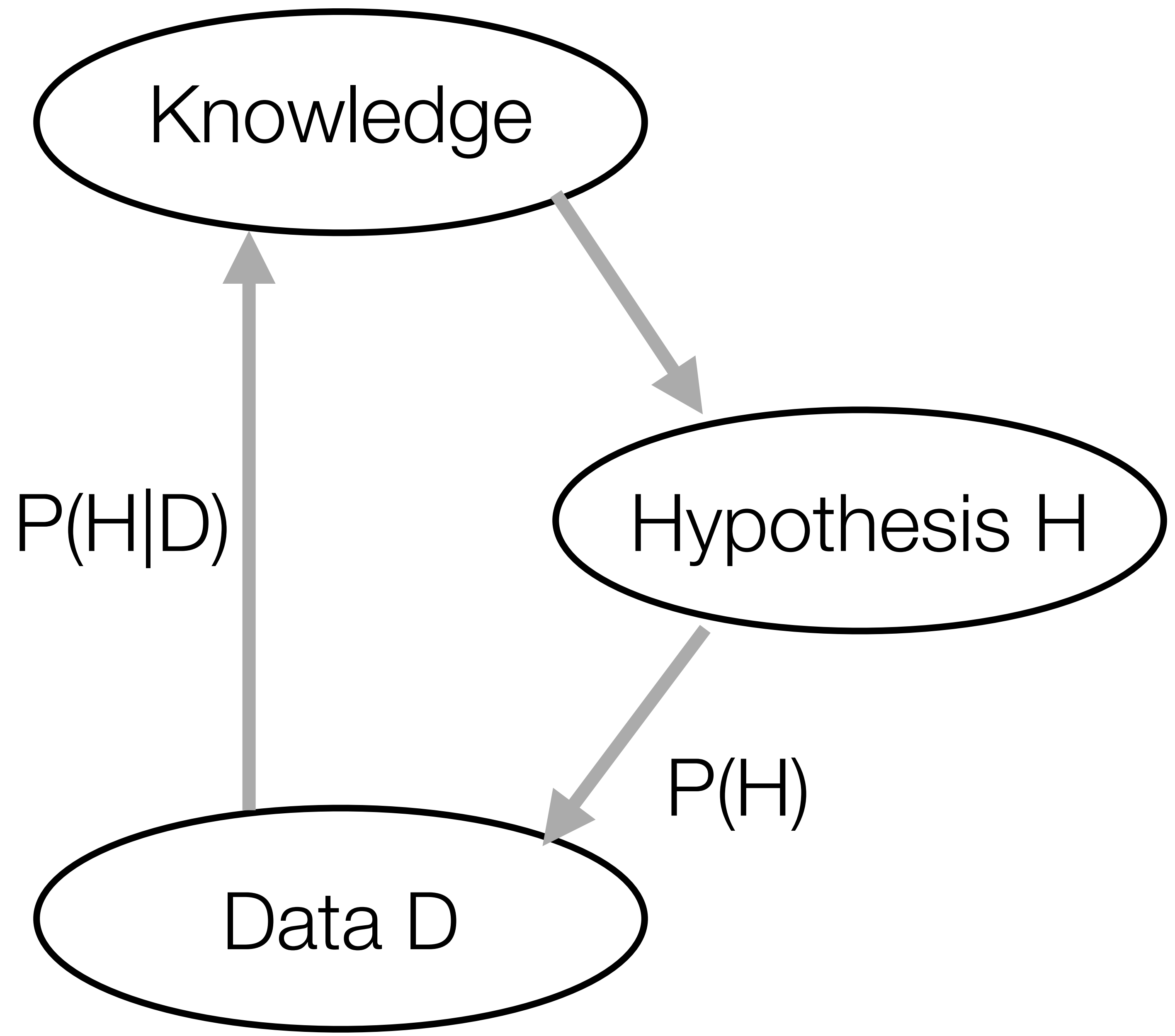
John Maynard Keynes:

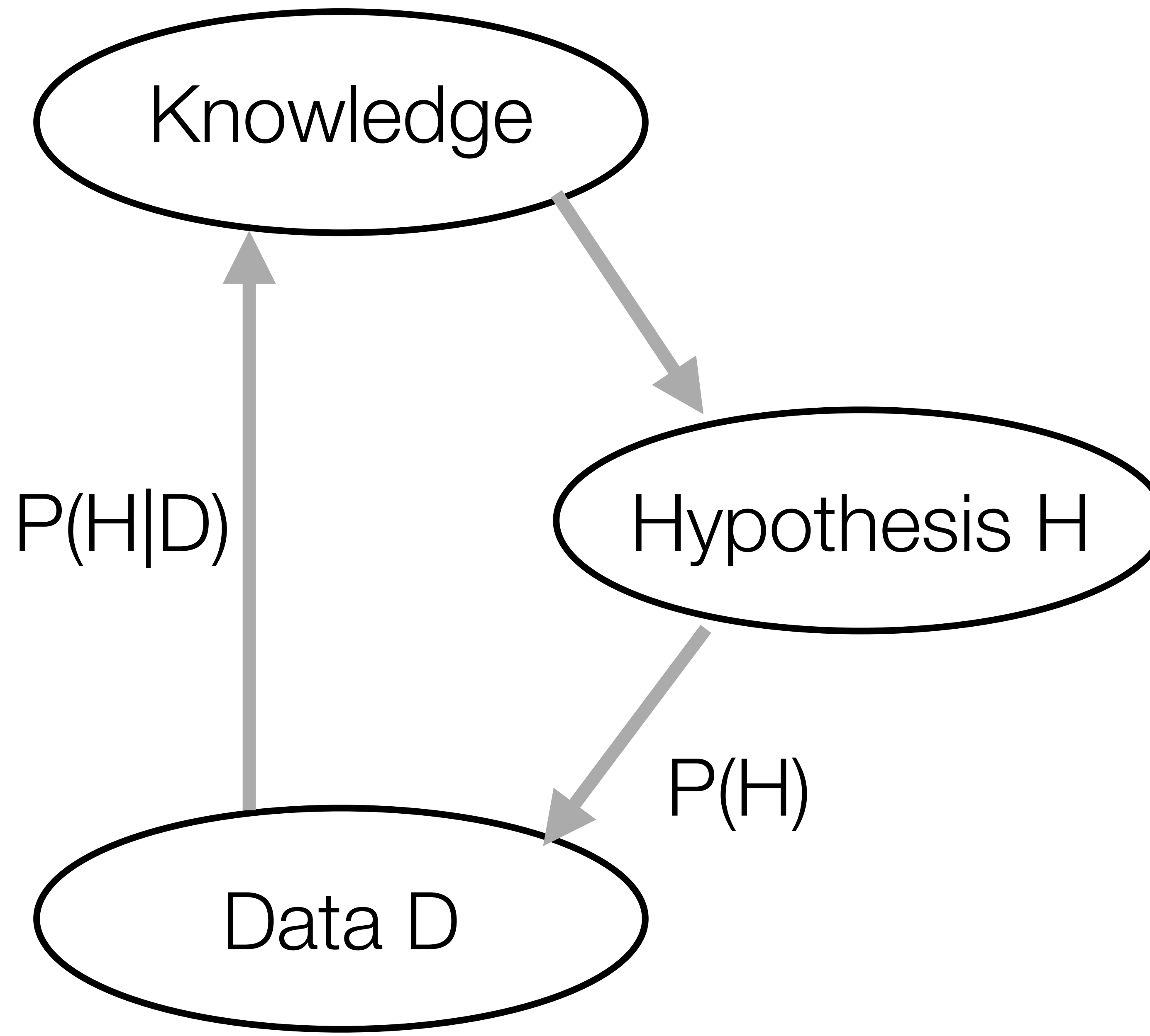
"In the long run, we are all dead"

Statistics as learning from data



Statistics as learning from data

- We almost always start with some prior knowledge, which leads us to test a hypothesis
  - Perform the PSA test
- We generally have some idea of what to expect
  - e.g. P(disease in next 10 years)=0.058
- We update our knowledge based on the data using Bayes' rule
  - P(disease|test result)=0.14



Dissecting Bayes' rule

$$P(A|B) = \frac{P(B|A)}{P(B)} * P(A)$$

Dissecting Bayes' rule

prior: how likely did we think A was before we collected data?

$$P(A|B) = \frac{P(B|A)}{P(B)} * P(A)$$

Dissecting Bayes' rule

posterior: how likely do we think A is after we collected data?

prior: how likely did we think A was before we collected data?

$$P(A|B) = \frac{P(B|A)}{P(B)} * P(A)$$

Dissecting Bayes' rule

posterior: how likely do we think A is after we collected data?

prior: how likely did we think A was before we collected data?

$$P(A|B) = \frac{P(B|A)}{P(B)} * P(A)$$

relative likelihood of the data given A, versus the overall likelihood of the data

Odds

- A ratio expressing the likelihood of something happening relative to not happening

$$odds = \frac{P(A)}{P(\sim A)}$$

  - 1/1: “even odds”

- Example: What are the odds of rolling a six using a one-sided die?

$$odds\, in\, favor = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5} \qquad odds\, against = \frac{\frac{5}{6}}{\frac{1}{6}} = \frac{5}{1}$$

Bayesian odds

$$prior\,odds = \frac{P(A)}{P(\sim A)} \qquad prior\,odds = \frac{0.058}{1 - 0.058} = 0.061$$

$$posterior\,odds = \frac{P(A|B)}{P(\sim A|B)} \qquad posterior\,odds = \frac{0.14}{0.86} = 0.16$$

$$likelihood\,ratio = \frac{posterior\,odds}{prior\,odds} = 2.62$$

Summary

- Conditional probabilities allow to express the likelihood of some event, given some other event
- The statistical concept of independence revolves around whether one variable provides information about the value of another
- Bayes' theorem provides us with the means to invert conditional probabilities